GLOBAL FLOW FIELD FOR STEADY, INCOMPRESSIBLE, SEPARATED FLOW PAST TWO-DIMENSIONAL BODIES

L. A. Kozhuro

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The fundamental problem in the theory of steady separated flow at high Reynolds numbers Re is the determination of the global flow field described by viscous flow equations. From general considerations and experimental data, it is clear that the boundary of the separated flow region in the ideal flow model for flows at large but finite Re is a surface where a discontinuity in tangential velocity occurs and the total pressure (Bernoulli's constant) undergoes a jump.

Furthermore, it is known [1, 2] that with fairly general assumptions, the limiting case of high Reynolds number two-dimensional flow with closed streamlines has a constant vorticity. Hence ideal fluid flow separation on two-dimensional body, which could serve as a first approximation for the analysis of viscous separation, should have constant vorticity in the separated region and a jump in the total pressure at its boundary.

An attempt to obtain such a solution for symmetric flow past a two-dimensional wedge was made in [4] using Batchelor's [3] model. However, since the problem was solved approximately, the solution was incorrect near the base of the wedge.

The majority of existing models for flow separation of an ideal fluid have a total pressure jump at the boundary of the separation region but the fluid within the region is stationary [5]. Solutions were recently obtained for Euler equations with constant vorticity in the separated region but without discontinuity in total pressure at its boundary [6].

The present paper deals with the flow past a flat plate placed normal to the flow. Flow separation is analyzed using Euler equations with constant vorticity in the separated region and a jump in total pressure at its boundary. The closure of the separation region is, apparently, not possible in the general case without introducing unrealistic singularities as in the case of flow with static fluid in the separation region of a cavity, with the pressure not exceeding the free stream pressure. But, as in the case of a jet, it is possible to expect that the flow as a whole depends weakly on the nature of the singularity in the reattachment region of the separation area. Therefore, Ryabushinskii's method is used in the present paper to analyze the closure of the separation region since it is the most convenient scheme for numerical computations.

The flow in the complex plane z = x + iy is described in Fig. 1a. In view of symmetry, only the upper half plane is considered. The flow also is symmetric relative to the y axis. It is necessary to determine the boundary of the separation region BE (curve L) and Stokes' function ψ satisfying the following conditions:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\begin{cases}0, & z\overline{\in} S,\\ -\omega, & z\in S;\end{cases} \tag{1}$$

$$\left(\frac{\partial\psi}{\partial n}\right)_e^2-\left(\frac{\partial\psi}{\partial n}\right)_i^2=\Delta,\quad z\in L,$$

$$\psi|_{ABEF}=0,\ \partial\psi/\partial y\to 1,\ z\to\infty, \tag{2}$$

where S is the separation region; ω is the vorticity in the region S; indices e and i denote outer and inner values of the derivatives of the stream function along the normal n to the boundary S; $\Delta = v_e^2 - v_i^2$ is the nondimensional double jump in the total pressure at the boundary of the separation region, normalized with respect to free stream paramters; $\mathbf{v} = (\partial\psi/\partial y, -\partial\psi/\partial x)$ is the flow velocity. The half-width of the flat plate and the velocity at infinity are assumed to be unity.

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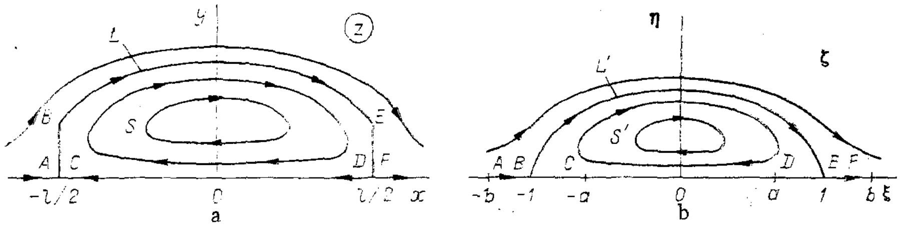

Fig. 1

Three parameters are included in the problem; Δ, ω, and l, the separation length. As shown by numerical results, the values of two of them can be arbitrarily specified, i.e., there is a two-parameter family of solutions to the given problem.

In order to satisfy no-slip conditions at the flat plate surface, the flow field is mapped on the upper half-plane of the complex variable $\zeta = \xi + i\eta$ (see Fig. 1b). The transformation has the form

$$z = \frac{c}{b}\int_0^{\zeta} \frac{1-\zeta^2}{\sqrt{(a^2-\zeta^2)(b^2-\zeta^2)}}\,d\zeta. \tag{3}$$

The real quantities a, b, and c are determined from the system of equations

$$l = 2c\left[\frac{1-b^2}{b^2}K(k) + E(k)\right], \quad K(k') - b^2E(k') = 0,$$

$$c\left[\frac{1-b^2}{b^2}F(\alpha, k') + K(k') - E(k') - F(\alpha', k') + E(\alpha', k')\right] = 1,$$

where $k = a/b$, $k' = \sqrt{1-k^2}$, $\sin\alpha = \sqrt{1-a^2}/k'$; $\sin\alpha' = \sqrt{b^2-1}/(bk')$; K(k) and E(k) are complete and F(α, k) and E(α, K) are partial, normal, Legendre elliptic integrals of the first and the second kind.

The stream function satisfying Eq. (1), the no-slip condition at the flat plate surface, and the free stream conditions can be represented in the form

$$\psi(\xi_0, \eta_0) = \frac{c}{b}\eta_0 - \frac{\omega}{2\pi}\iint_{S'}\left|\frac{dz}{d\zeta}\right|^2 \ln\left|\frac{\zeta_0-\zeta}{\zeta_0-\bar{\zeta}}\right| d\xi d\eta - \frac{1}{2\pi}\int_{L'}\gamma(\xi, \eta)\ln\left|\frac{\zeta_0-\zeta}{\zeta_0-\bar{\zeta}}\right| dl, \tag{4}$$

whre S' and L' are the maps of the regions S and L respectively; dl is the elemental length of the curve L'; $\gamma(\xi, \eta)$ is the strength of the vortex layer at the edge of the separation region in the ζ plane, $\zeta_0 = \xi_0 + i\eta_0$.

The condition (2) in the ζ plane takes the form

$$\left(\frac{\partial\psi}{\partial n}\right)_e^2 - \left(\frac{\partial\psi}{\partial n}\right)_i^2 = \Delta\left|\frac{dz}{d\zeta}\right|^2, \quad \zeta \in L'$$

and is expressed in the following manner [7]:

$$\gamma(\xi_0, \eta_0) = -\frac{\Delta}{4\pi\sqrt{1+\eta_0'^2}}\left|\frac{dz}{d\zeta}\right|^2_{\zeta=\zeta_0}\left\{\frac{2\pi c}{b} - \omega\iint_{S'}\left|\frac{dz}{d\zeta}\right|^2 \times\right. \tag{5}$$

$$\times\left[\frac{(\xi-\xi_0)\eta_0' - \eta + \eta_0}{(\xi-\xi_0)^2 + (\eta-\eta_0)^2} - \frac{(\xi-\xi_0)\eta_0' + \eta + \eta_0}{(\xi-\xi_0)^2 + (\eta+\eta_0)^2}\right] d\xi d\eta - \int_{L'}\gamma(\xi, \eta)\times$$

$$\left.\times\left[\frac{(\xi-\xi_0)\eta_0' - \eta + \eta_0}{(\xi-\xi_0)^2 + (\eta-\eta_0)^2} - \frac{(\xi-\xi_0)\eta_0' + \eta + \eta_0}{(\xi-\xi_0)^2 + (\eta+\eta_0)^2}\right] dl\right\}^{-1}, \quad \eta_0' = \frac{d\eta_0}{d\xi_0}, \quad \zeta_0 \in L'.$$

Vorticity ω is determined from the condition of finite velocity at the sharp flat plate edge. In the ζ plane, this condition means $\partial\psi/\partial\eta|_{\zeta=-1} = 0$. Using Eq. (4), we get

$$\omega = -\left[\frac{\pi c}{b} + \int_{L'}\gamma(\xi, \eta)\frac{\eta}{(1+\xi)^2+\eta^2}dl\right]\left[\iint_{S'}\left|\frac{dz}{d\zeta}\right|^2 \frac{\eta}{(1+\xi)^2+\eta^2}d\xi d\eta\right]^{-1}. \tag{6}$$

The condition $\psi(\xi, \eta) = 0$, $(\xi, \eta) \in L'$, with the help of Eqs. (4) and (5) forms a system of integral equations for the determination of the unknown curve L' and the strength of the vortex layer γ located on it. Vorticity ω in these equations is determined from Eq. (6).

The condition of finite velocity at the flat plate edge means that the streamline $\psi = 0$ at the point where it separates from the flat plate B (see Fig. 1a) is perpendicular to the x axis. It follows from the properties of the transformation (3) that in the mapping of the point B on the ζ plane, the curve L' should be perpendicular to ξ axis. The vortex strength at the point B in the z plane is equal to $-(v_e - v_i) = \gamma(-1)/|dz/d\zeta|_{\zeta=-1}$. Hence, in view of the finite value of the velocity at the point B, $\gamma(-1) = 0$.

The resulting system of integral equations was numerically solved using iterative technique. Iterations for the strength of the vortex layer γ were carried out, as in [7], by computing the right-hand side of Eq. (5) with the previous approximation. The following expression was used in the succeeding iteration for the strength of the vortex layer

$$\gamma_{i+1} = \gamma_i + k(\xi)[\gamma_0(\xi, \gamma_i, \eta_i) - \gamma_i], \ \xi \in [-1, 1],$$

where $\eta_i = \eta_i(\xi)$ and $\gamma_i = \gamma_i(\xi)$ are the preceding approximations for the curve L' and the strength of the vortex layer located on it; $\gamma_0(\xi, \gamma_i, \eta_i)$ is the right hand side of Eq. (5) computed with these approximations; $k(\xi)$ is a positive quantity not exceeding one. This function was chosen on the basis of numerical experiment. The iterative procedure converged quite well with a monotonic change in the function $k(\xi)$ from $k(-1) = 0.25$ to $k(0) = 1$.

The following iterative scheme was used for the boundary of the vortex flow region:

$$\eta_{i+1} = \eta_i + \frac{2\psi_0(\xi, \eta_i, \gamma_i)}{\gamma_{i+1} + \Delta\left|\frac{dz}{d\zeta}\right|^2 \Big/ \gamma_{i+1}} \sqrt{1 + \eta_i'^2}, \quad \xi \in [-1, 1],$$

$\eta_{i+1} = \eta_{i+1}(\xi)$ is the succeeding approximation for the shape of the curve L'; $\psi_0(\xi, \eta_i, \gamma_i)$ is the right-hand side of Eq. (4), computed with functions η_i and γ_i.

This relation is an approximate expression for the first iteration in Newton's iterative scheme for the equation $\psi(\xi, \eta) = 0$ for a fixed value of ξ. Here $\eta_i(\xi)$ is used as the zeroth approximation for η and the outer value of the derivative $\partial\psi/\partial\eta$ is used:

$$\left(\frac{\partial\psi}{\partial\eta}\right)_e = v_{\xi_e} \approx \frac{v_e}{\sqrt{1 + \eta'^2}} \approx -\frac{\gamma + \Delta\left|\frac{dz}{d\zeta}\right|^2 \Big/ \gamma}{2\sqrt{1 + \eta'^2}}.$$

The last equation follows from relations $v_e^2 - v_i^2 \simeq \Delta|dz/d\zeta|^2$ and $v_e - v_i \simeq -\gamma$, whose accuracy increases with the convergence of the iteration. The method adopted in the present paper to correct the boundary of the vortex flow was considerably more effective than that used in [7].

Functions $\eta(\xi)$ and $\gamma(\xi)$ were assumed to be even functions in ξ and were specified in the form of natural cubic splines [8] for the parameter $t \in [0, 2]$. The function $\xi(t)$ was specified as follows:

$$\xi(t) = \begin{cases} t^2 - 1, & t \in [0, 1], \\ 1 - (t - 2)^2, & t \in [1, 2]. \end{cases}$$

Adaptive quadrature program [8] was used in computing integrals. The logarithmic and algebraic singularities in the integrand were separated.

The shape of the separation region where there is no total pressure jump ($\Delta = 0$) is shown in Fig. 2 for different values of the separation length. The half width of the vortex flow region is assumed one. Figure 2 also shows the contour of the region with constant vorticity in the plane potential flow, viz., the two-dimensional analog of Hill's vortex obtained in

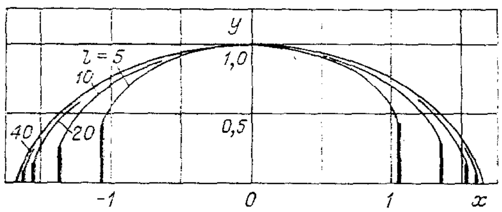

Fig. 2

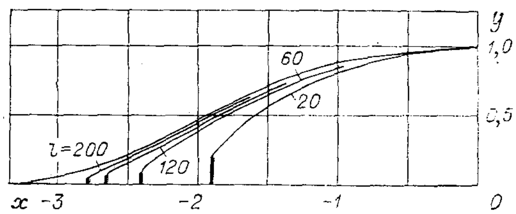

Fig. 3

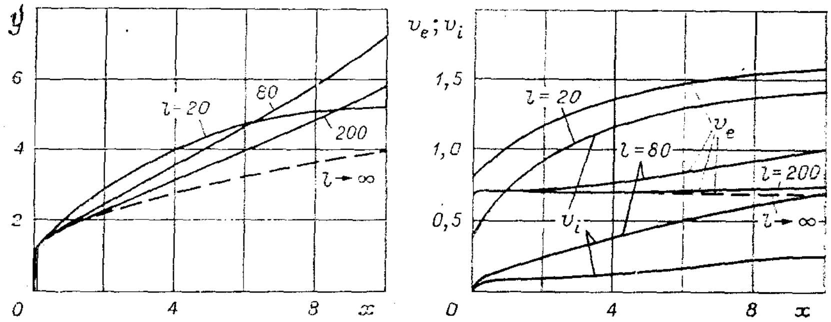

Fig. 4

[9, 10]. It is seen that this flow is the limiting case for the resulting family of separated flows.

The contours of separation region when $\Delta = 0.5$ are shown in Fig. 3. The contour of the region with constant vorticity with the same jump in total pressure corresponding to the family of degenerate vortex flows [7, 11] is also shown in Fig. 3. Even in this case convergence is quite obvious.

Thus, computed results give basis to state that two-parameter family of separated flows near two-dimensional bodies has, as its limiting case, a one-parameter family of the abovementioned degenerate vortex flows as the length of the separation zone tends to infinity in terms of the dimensions of the separation zone.

The shape of the separation region near the body is shown in Fig. 4 for $\Delta = 0.5$. The outer and inner velocity profiles at the edge of the separation zone are also shown here. The dashed lines indicate flow past a flat plate using Kirchhoff's method with the free stream velocity equal to $\sqrt{\Delta}$. It is seen that with unbounded increase in the separation length, the flow near the body approaches Kirchhoff's flow with a characteristic velocity corresponding to the value of the jump in the total pressure.

The values of flat plate skin friction coefficient c_x for the two one-parameter family of flows shown in Figs. 2 and 3 are given in Fig. 5. As the separation length tends to infinity, the skin friction coefficient of the flat plate tends to zero when $\Delta = 0$ and when $\Delta = 0.5$, to the value $2\pi\Delta/(\pi + 4) \approx 0.44$, corresponding to the flat plate skin friction coefficient computed by Kirchhoff's method for the free stream velocity equal to $\sqrt{\Delta}$.

The values of the parameter $\Omega = -\omega l/2$ from [10] are also given in Fig. 5. A value $\Delta = 0.5$ (shown by an asterisk in Fig. 5) has been obtained in [11] for the degenerate vortex flow with $\Omega = 9.5$ which agrees well with the present results as $l \to \infty$. The limiting value of this parameter as $l \to \infty$ and $\Delta = 0$ ($\Omega = 6.5$), corresponding to the two-dimensional analog of Hill's spherical vortex, is significantly different from the result given in [10, 11] ($\Omega = 7.063$) and agrees well with the result in [12] ($\Omega = 6.469$).

Different ideal flow schemes were suggested as examples of the global picture of steady separated viscous flow past a body at high Re. Based on Prandtl's boundary layer assumptions and known properties of boundary layer and mixing layer solutions, Prandtl [13] and then Imai [14] assumed that the jet flow past a body using Kirchhoff's scheme can be considered as a solution to Navier–Stokes equations in the limiting case of zero viscosity. This scheme for separated flow has an infinite separated region and cannot be used for describing the global flow field of separation at large, but finite Re, since the dimensions of the separated zone should be finite.

In the separation model suggested in [15, 16] and discussed in detail in [17], the separation length is proportional to Re and the width to the square root of Re. The fluid in the separated zone is stationary to the first approximation. In the limit as Re becomes infinitely large, the flow in the neighborhood of the body, as in the previous model, approaches the flow in the Kirchhoff scheme. However, as shown in [18, 19], conservation of energy is not satisfied for this scheme: To realize such a flow it is necessary to have special boundary conditions inside the separation zone, viz., an extremely strong dissipator which is absent in the real problem of flow past a body. Furthermore, stationary flow within the closed separation region is not possible in this model [16]. Hence the stationary character of closure of the separation region justifying the use of this model has been suggested in [16].

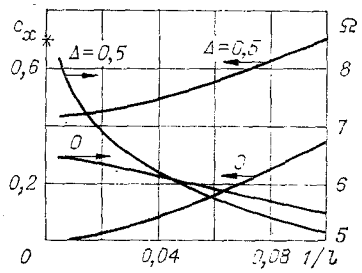

Fig. 5

Separation model in [3] has a separation region with dimensions of the order of the body. In the separation region, the fluid in the two-dimensional flow case has a constant vorticity and the total pressure undergoes a jump at its boundary. However, not a single flow has been realized so far using this scheme for separated flow (for solving the problem with the exact formulation).

In the separation model suggested in [19-21], the size of the separation region unboundedly increases with Re, while the transverse and streamwise dimensions have the same order of magnitude. The flow within the separated zone as in Batchelor's model has constant vorticity in the two dimensional case. The total pressure undergoes a jump at the edge of the region and its value tends to zero as Re tends to infinity. In terms of body dimensions, the flow is close to the Kirchhoff model but the characteristic velocity near the body tends to zero with increase in Re. The skin friction coefficient also tends to zero correspondingly. In the limiting case of infinitely large Re, the body is reduced to a point in terms of the dimensions of the separated zone and the global flow field in the axisymmetric case is Hill's vortex and in the two-dimensional case, its two-dimensional analog [9, 10]. The family of limiting separated flows with a dissipator inside the separated region in the two-dimensional and axisymmetric cases are given in [7, 11, 12, and 22].

The family of separated flow solutions to Euler equations obtained in the present paper fully agrees with G. I. Taganov's separated flow model: The global flow field of separated flow approaches the corresponding degenerate vortex flow as the body dimensions tends to zero in the scale of the separated zone while the flow in the neighborhood of the body approaches Kirchhoff flow with a characteristic velocity corresponding to the jump in the total pressure. Thus, a family of ideal fluid flows has been obtained on the basis of which it is possible to construct stationary viscous separated flow at large Re.

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EFFECT OF PERIODIC BOTTOM ROUGHNESS ON GRAVITATIONAL WAVES IN A LIQUID

V. I. Merkulov, A. P. Mikhailov, and S. O. Savchenko

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The effect of a rigid bottom of periodic form on small periodic oscillations of the free surface of a liquid is considered with the assumption of low amplitude roughness. The methodologically most significant study in this direction, [1], will be utilized. In [1] the steady-state problem for flow over an arbitrarily rough bottom was studied. Other studies have recently appeared on small free oscillations above a rough bottom. Essentially these have considered the effect of underwater obstacles and cavities on surface waves in the shallow-water approximation (for example, [2], [3]). Liquid oscillations in a layer of arbitrary depth slowly varying with length were considered in [4]. However, these results cannot be applied to the study of resonant interaction of gravitational waves with a periodically curved bottom.

1. We will consider a plane layer of nonviscous incompressible liquid extending infinitely in the x-direction and lying in a gravitational field above a periodically rough bottom (Fig. 1). The form of the bottom is specified by the function $a\lambda(x)$, where a is the amplitude of the bottom roughness, H is the mean height of the layer. The studies were carried out using dimensionless variables, the wavelength of the bottom period being taken equal to 2π. Only periodic small free oscillations of the layer at rest were studied, with the conditions $a \ll 1$, $a/H \ll 1$.

Let φ be the velocity potential. Then if the liquid density is much greater than the density of air and $(\nabla\varphi)^2$ is small, a linearized boundary problem for small free oscillations on a rough bottom is known (see, for example, [5, 6]):

$$\Delta\varphi = 0,\ \frac{\partial\varphi}{\partial \mathbf{n}} = 0\Big|_{y=a\lambda(x)},\ \frac{\partial\varphi}{\partial y} = -\frac{1}{g}\frac{\partial^2\varphi}{\partial t^2}\Big|_{y=H} \tag{1.1}$$

or for a single harmonic $\varphi = e^{i\omega t}u(x, y)$

$$\Delta u = 0,\quad \frac{\partial u}{\partial \mathbf{n}} = 0\Big|_{y=a\lambda(x)},\quad \frac{\partial u}{\partial y} = \frac{\omega^2}{g}u\Big|_{y=H}, \tag{1.2}$$

where $\mathbf{n}$ is a unit vector normal to the line of the bottom.

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